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THE EFFECT OF MICRODAMAGE ON FATIGUE-CRACK GROWTH†

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The theory of fatigue-crack growth, based on a synthesis of fracture mechanics and continuum mechanics of microdamage accumulation, is applied to the problem of crack growth under cyclic loading, taking into account the plastic deformations in the tip zones. The model of a thin plastic zone, which is a region of considerable plastic deformations at the crack tip, is supplemented by taking into account the effect of microdamage on the value of the specific fracture work and the limit stresses in the tip zone. Governing equations which describe fatigue-crack growth taking these factors into account are derived. The effect of the material characteristics and the load parameters on the growth rate and the distribution of microdamage in the tip zone and on its extensions is investigated by a computational experiment. Particular attention is given to the initial stage when crack growth may occur abruptly and the growth rate depends substantially on the initial conditions. © 1997 Elsevier Science Ltd. All rights reserved.

Crack growth under cyclic loading is often accompanied by plastic deformation. This is observed, in particular, at high cyclic stress levels, when fracture is preceded by a relatively low number of cycles (usually of the order of 10^3–10^4). In such cases we speak of low-cycle fatigue. Plastic deformations may also play a considerable role for short cracks, the length of which is comparable with the dimensions of the plastic-deformation zone. A review of the experimental evidence and mathematical models used to describe these phenomena can be found in [1–7].

The mechanics of fatigue-crack growth is, essentially, a further development of fracture mechanics, which mainly considers the behaviour of cracked bodies under monotonically increasing loads. Fatigue-crack growth is a result of the interaction of two mechanisms: the accumulation of scattered damage in the material as a result of cyclic stresses, and the overall energy balance in the system of the cracked body with cracks and the load or the loading device. The synthesis of fracture mechanics and damage accumulation mechanics enables fatigue fracture and allied phenomena to be described quantitatively.

In non-linear fracture mechanics the model of a thin plastic zone [3, 5, 7] is one of the simplest. In this model all plastic effects are concentrated in tip zones of finite length that are infinitesimally thin. The limit fracture stresses within the tip zones are usually taken as a constant of the material, like the yield point for tensile stress σ_γ, sometimes identifying the limit stress with σ_γ or $\sqrt{(3)}\sigma_\gamma$. Outside the plastic zone the material is assumed to be linearly elastic. The length of the plastic zone is determined from the condition for removing singularities when joined with the elastic solution. Nevertheless, the plastic regions, determined from the theory of plastic flow, have dimensions of the same order as for the extension of the crack, and in the transverse direction [5]. Despite this rough scheme, the model of a thin plastic zone give satisfactory quantitative results in the case of quasi-brittle fracture. This is the reason it is widely used in non-linear fracture mechanics, also for predicting subcritical crack growth under cyclic loading, when interpreting experimental data and when estimating the crack resistance of structural materials [1, 2, 5]. A number of extensions of the thin plastic zone model have been proposed, including taking plastic hardening and softening into account [2].

Under cyclic loading microdamage occurs in the tip zones, i.e. microcracks and micropores, which produce a change in the macroscopic properties of the material. A reduction in the specific fracture work due to the effect of microdamage is the main mechanism for the motion of the fatigue crack front. An increase in the specific fracture work due to the screening effect of micropores, and of deformation hardening and deformation (acquired) anisotropy has also been observed. Another group of defects is the change in deformative properties, which can manifest themselves both in the form of hardening and softening. In particular, a reduction in the material stiffness leads to a reduction in the stresses in the neighbourhood of the crack tips, which produces a change in the rate of microdamage accumulation [10]. As a result the value of the work which must be expanded in advancing the fatigue crack is increased.

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To describe fatigue-crack growth, it is necessary to supplement the plastic-zone model by relations which describe the microdamage accumulation, and also the effect of this damage on the mechanical properties of the material. This was done for the first time in a quasi-steady approximation [11], where, when a plastic zone is developing, a power relationship was obtained for the fatigue-crack growth rate as a function of the stress intensity factor with an exponent close to two. A more detailed analysis was carried out in [12] using a numerical method. A threshold-power relationship was used in this case to describe the microdamage accumulation, and the effect of microdamage on the specific fracture work was taken into account; the effect of microdamage on the deformation characteristics of the material was ignored. In particular, it was assumed that the limit stress remains constant over the whole tip zone. In this paper we remove this constraint, which enables us to describe more fully the phenomena and processes which accompany fatigue-crack growth.

1. Following the approach described earlier [8], we will treat the system consisting of the cracked body and the load as a mechanical system with unilateral constraints. For simplicity, we will consider a single-parameter crack, whose dimensions are specified by the length parameter a, for example, the half-length of the crack in the Griffith problem (Fig. 1). We will assume the crack to be irreversible, i.e. we will assume that the variation in its parameter $\delta a \geqslant 0$. For a system with unilateral constraints the equilibrium condition has the form

$$\delta A \leqslant 0 \tag{1.1}$$

As it applies to quasi-static problems in fracture mechanics, all the mixed states of equilibrium can be compared in the usual sense (with respect to the generalized Lagrange coordinates). Hence, when investigating the stability of a system consisting of the cracked body and the load, condition (1.1) can be replaced by the following

$$\delta_G A \leqslant 0 \tag{1.2}$$

The subscript on $\delta_G A$ denotes that the perturbed state differs solely in the displacements which describe the crack (these displacements are called [8] the generalized Griffith coordinates). We will represent the left-hand side of (1.2) in the form

$$\delta_G A = \delta_G A_e + \delta_G A_i + \delta_G A_f \tag{1.3}$$

where $\delta_G A_e$ and $\delta_G A_i$ are the virtual works of the external and internal forces, respectively, and $\delta_G A_f$ is the virtual fracture work. We can write the right-hand side of (1.3) in terms of generalized forces as follows:

$$\delta_G A_e + \delta_G A_i = G\delta a, \quad \delta_G A_f = -\Gamma \delta a \tag{1.4}$$

The identities (1.4) introduce two types of generalized forces: a generalized driving force G and a generalized resistance force Γ. In linear fracture mechanics the generalized force G corresponds to the intensity of the released energy G or the J-integral, while the generalized resistance force represents the corresponding crack-resistance characteristics G_C or J_C [5, 7].

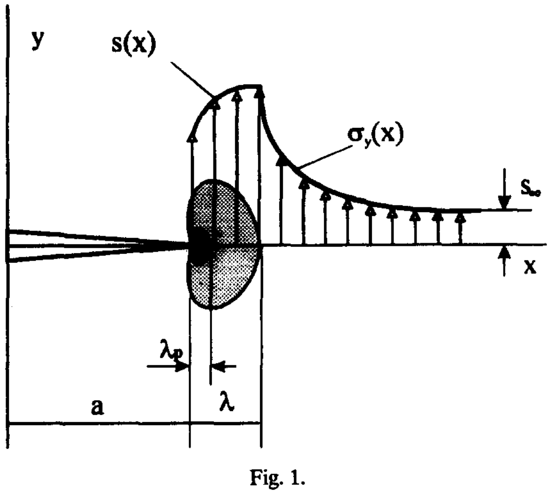

Fig. 1.

In single-parameter problems, condition (1.2) takes the form

$$G \leqslant \Gamma \tag{1.5}$$

Depending on the signs in (1.2) and (1.5) we will distinguish the following states of the system consisting of the cracked body and the load. If for any $\delta a > 0$ we have $\delta A < 0$, i.e. $G < \Gamma$, the state is said to be a subequilibrium state. This state is obviously stable. If $\delta a = 0$, i.e. $G = \Gamma$, the state is said to be an equilibrium state (with respect to the generalized Griffith coordinates). This state can be stable or neutral depending on the behaviour of the difference $G-\Gamma$ in the neighbourhood of the equilibrium state investigated. In particular, it is stable when $\partial G/\partial a < \partial\Gamma/\partial a$. Finally, when $\partial A > 0$, i.e. $G = \Gamma$, the state is unstable; we will call this a non-equilibrium state.

Conclusions regarding stability (instability) in this case are almost obvious from the physical point of view; a rigorous analysis, however, has only been carried out as it applies to potential systems [13]. Nevertheless, it has been shown in [8, 14, 15], that relations (1.2) and (1.5) do not contradict the generally accepted ideas of fracture mechanics, and also experimental facts. These relations have been used repeatedly [5, 7] in implicit form or in the form of energy-balance relations. Relations of the form (1.2) and (1.5) have been applied to fatigue cracks [10–12, 15, 16], and also to problems of dynamic crack propagation (in conjunction with d'Alembert's principle) [17].

2. Consider a cleavage crack in an unbounded elastoplastic medium under plane stress (Fig. 1). We will denote the half-length of the crack by a, the length of the plastic zone for monotonic loading by λ, and the stress applied at infinity by σ_∞. We will denote the stress $\sigma_y(x, y)$ along the extension of the crack $|x| \geqslant a, y = 0$ simply by $\sigma_y(x)$, and we will denote the limit stress $\sigma_y(x, 0)$ in the plastic zone $a \leqslant |x| \leqslant a + \lambda$ by $s(x)$.

To determine the stress, deformation and displacement fields in a cracked body for an arbitrary stress distribution $s(x)$ in the tip zones it is best to use available analytic solutions for the case s = const. These solutions have been obtained by a number of investigators [3, 4]. Different analytical methods lead to identical though superficially different results. We will use those results, which are more convenient for further calculations.

The stress fields in a cracked body can be conveniently represented using the complex Westergard potential $Z(z)$ [6]. Then

$$\sigma_x = \operatorname{Re} Z - y \operatorname{Im} Z', \quad \sigma_y = \operatorname{Re} Z + y \operatorname{Im} Z' \tag{2.1}$$

$$\tau_{xy} = -y \operatorname{Re} Z', \quad Z' \equiv dZ/dz, \quad z = x + iy$$

For two single forces applied normal to the crack edges at the point $x = \xi(|\xi| < a)$, we have

$$Z(z) = \frac{(a^2 - \xi^2)^{1/2}}{\pi(z - \xi)(z^2 - a^2)^{1/2}} \tag{2.2}$$

In the classical problem of a thin tip zone, we will assume that the critical normal stress $\sigma_y = 0$ when $|\xi| < a$ and $\sigma_y = s$ = const when $a \leqslant |\xi| \leqslant a + \lambda \equiv b$. The application of potential (2.2) to this problem yields

$$Z(z) = \frac{z}{(z^2 - b^2)^{1/2}}\left[\sigma_\infty - \frac{2s}{\pi}\arccos\frac{a}{b}\right] + \frac{2s}{\pi}\operatorname{arctg}\left[\frac{a}{z}\left(\frac{(z^2 - b^2)}{b^2 - a^2}\right)^{1/2}\right] \tag{2.3}$$

Suppose $s(x)$ is a differentiable function over the whole region $a \leqslant |x| \leqslant b$. Then, taking (2.3) into account and also the condition for the stresses to be non-singular when $x = b, y = 0$, which gives an equation for determining the length of the boundary zone λ (this equation is a generalization of a well-known formula [4, 7])

$$s(a)\arccos\frac{a}{a+\lambda} + \int_a^{a+\lambda} \frac{ds(\xi)}{d\xi}\arccos\frac{\xi}{a+\lambda}\,d\xi = \frac{\pi\sigma_\infty}{2} \tag{2.4}$$

we obtain

$$Z(z)=\frac{2s(a)}{\pi}\operatorname{arctg}\left[\frac{a}{z}\left(\frac{(z^2-b^2)}{b^2-a^2}\right)^{1/2}\right]+\frac{2}{\pi}\int_a^b\frac{ds(\xi)}{d\xi}\operatorname{arctg}\left[\frac{\xi}{z}\left(\frac{(z^2-b^2)}{b^2-\xi^2}\right)^{1/2}\right]d\xi \tag{2.5}$$

For the displacements $v\ (x, 0)$ at the crack edges and extension of the crack within the tip zone, it is more convenient to start from the formula [3]

$$v(x,\ 0)=cs[(x-a)\Gamma(b,\ x,\ a)-(x+a)\Gamma(a,\ x,\ -a)] \tag{2.6}$$

where $c = 1/(\pi E)$ for a plane-stressed state and $c = (1-\nu^2)/(\pi E)$ for plane deformation (E is Young's modulus and ν is Poisson's ratio). The kernel has the form

$$\Gamma(b,\ x,\ \xi)=\ln\frac{b^2-x\xi-[(b^2-x^2)(b^2-\xi^2)]^{1/2}}{b^2-x\xi+[(b^2-x^2)(b^2-\xi^2)]^{1/2}} \tag{2.7}$$

If $s = s(x)$, then instead of (2.6) we arrive at a formula which is similar in structure to (2.5)

$$v(x,\ 0)=cs(a)[(x-a)\Gamma(b,\ x,\ a)-(x+a)\Gamma(a,\ x,\ -a)]+$$
$$+c\int_a^b\frac{ds(\xi)}{d\xi}[(x-\xi)\Gamma(b,\ x,\ \xi)-(x+\xi)\Gamma(a,\ x,\ -\xi)]d\xi \tag{2.8}$$

3. Suppose the stresses σ_∞, given at infinity, very cyclicly with time. We will denote the number of cycles, taken as the independent variable, by N; $\sigma_\infty^{\max}(N)$ and N; $\sigma_\infty^{\min}(N)$ denote the extremal stresses of the cycle. In addition, we introduce the range of the stresses within a cycle $\Delta\sigma_\infty(N) = \sigma_\infty^{\max}(N) - \sigma_\infty^{\min}(N)$ and the symmetry characteristic of the cycle $R(N) = \sigma_\infty^{\min}(N)/\sigma_\infty^{\max}(N)$. In order not to introduce complications due to the effect of crack closure [18], we will assume that the specified nominal stresses remain tensile stresses within a cycle.

We will take condition (1.5) for a fatigue crack with length parameter $a(N)$ in the form

$$G[\sigma_\infty^{\max}(N),\ a(N)]\leqslant\Gamma[\psi(N),\ a(N)] \tag{3.1}$$

where $\psi(N)$ is a measure of the microdamage at the crack tip $|x| = a$. Here we assume that the generalized driving force G is independent of the microdamage, while the generalized resistance force Γ is independent of the applied stresses. The introduction of $\sigma_\infty^{\max}(N)$ into the left-hand side of (3.1) denotes that the stability condition is verified at the instant when the cleavage stresses reach a maximum within each cycle.

We will express the generalized force G in terms of the virtual work carried out by the stress σ_y when the crack tip moves from $x = a$ to $x = a + \delta a$. Here we take into account that simultaneously with a, the dimensions of the tip zone $\lambda(a)$ are also subject to variation. Referring the work done to one of the crack tips, we obtain

$$G\delta a=\int_a^{a+\delta a+\lambda+\delta\lambda}\sigma_y(2\delta v)dx=2\int_a^{a+\lambda}\sigma_y\delta v\,dx+O(\delta v^2)$$

whence we arrive at the formula [11, 14]

$$G=2\int_a^{a+\lambda}\sigma_y\frac{\partial v}{\partial a}dx \tag{3.2}$$

The right-hand side of (3.2) is superficially similar to the corresponding formula for the J-integral. However, the integrand contains $\partial v/\partial a$ rather than $\partial v/\partial x$, as in the existing formula [5, 7]. For stresses σ_∞ close to the yield point of the material, the difference between G and J may be fairly large [19].

The right-hand side of inequality (3.1) contains the generalized resistance force Γ, which depends on the measure of damage at $x = a$, i.e. on

$$\psi \equiv \omega\,[a(N), N] \tag{3.3}$$

Here the function $\omega(x, N)$ is similar to the measure of damage of the Kachanov–Rabotnov continuum model [20]. We will use the threshold-power accumulation law for this measure. Treating N as a continuous parameter, we postulate the equation

$$\frac{\partial \omega}{\partial N} = \left(\frac{\Delta\sigma_y - \Delta\sigma_{th}}{\sigma_\omega} \right)^m (1-\omega)^{-n} \tag{3.4}$$

with the material parameters σ_ω, $\Delta\sigma_{th}$, m and n. Here σ_ω is the stress, characterizing the resistance of the material to microdamage accumulation and $\Delta\sigma_{th}$ is the threshold value of this resistance; the exponent $m > 0$, $n \geqslant 0$. If the range $\Delta\sigma_\infty(N)$ of the applied stress is sufficiently small, we have $\Delta\sigma_y(x, N) < \Delta\sigma_{th}$ at least along part of the length $|x| \geqslant a$. We must then take $\partial\omega/\partial N = 0$ instead of (3.4).

Note that the range $\Delta\sigma_y(x, N)$ is defined as the difference between the values of $\sigma_y(x, N)$, corresponding to the applied stresses $\sigma_\infty^{\max}$, and the unloading stress $\sigma_\infty^{\max} - \sigma_\infty^{\min}$. We will henceforth assume that, within each cycle, the material behaves as an ideal elasto-plastic material with a yield point $s^+(x)$ in the case of tension, and $s^-(x)$ for compression, and Young's modulus E for unloading. As a result, the dimensions of the plastic zone vary during loading. Henceforth, we will draw a distinction between the dimensions λ for monotonic loading and the dimensions λ_p for cyclic loading. Obviously $\lambda_p \leqslant \lambda$. In particular, for $R = 0$ and $s = \text{const}$ for $|x| \geqslant a$ we obtain that $\lambda_p = \lambda/4$. When $s = s(x, N)$ the dimensions λ_p will be sought numerically for each cycle (or block of cycles) using (2.1), (2.4) and (2.5).

We will take into account the effect of microdamage on the properties of the material as follows. First, we will assume that the generalized resistance force is a decreasing function of ψ, for example

$$\Gamma = \gamma[1 - (\psi/\psi_*)^\alpha] \tag{3.5}$$

where γ is the specific fracture work for the undamaged material, ψ_* is a certain constant (for example, $\psi_* = 1$), while the constant $\alpha > 0$, for example $\alpha = 1$. Second, we will assume that the limit stress $s(x)$ depends on the local value of ω. Henceforth we will assume

$$s = s_0[1 - c_*\omega^\beta], \quad c_* = \text{const}, \quad \beta = \text{const} > 0 \tag{3.6}$$

where s_0 is the limit stress for the undamaged material. Generally speaking, we draw a distinction between s_0^+ and s_0^-. The choice of formulae (3.5) and (3.6), as well as the right-hand side in (3.4), is obviously fairly arbitrary. In particular, in (3.5) and (3.6) it is easy to introduce the effect of cyclic hardening or a combination of hardening and softening. A typical relationship, which describes cyclic hardening and subsequent softening has the form $\Gamma = \gamma(1 + \psi^{\alpha_1})$ when $0 \leqslant \psi \leqslant \psi_1$ and $\Gamma = \gamma(1 + \psi_1)^{\alpha_1} - (\psi - \psi_1)^{\alpha_2}$ when $\psi_1 \leqslant \psi \leqslant 1$.

The calculation of fatigue-crack growth and a parametric analysis are based on the stability condition (3.1) combined with the equation of the damage accumulation (3.4) and formulae (3.2), (3.5) and (3.6). Since the functions $\sigma_y(x)$ and $v(x)$ occurring in these formulae as well as the length of the cyclic plastic zone λ_p in turn are determined from (2.1), (2.4) and (2.8), further analysis can only be carried out numerically. Available software enables this calculation to be carried out. It includes modules for calculating the measure of damage and the stresses ahead of a crack (Eqs (2.1), (2.4), (2.5), (3.4), and (3.6)), the calculation of the generalized forces (formulae (3.2) and (3.5)), and also a module for checking condition (3.1), which defines the instant when the crack starts to grow and the balance of generalized forces as the crack front advances. By modifying individual modules one can replace Eqs (3.4)–(3.6) by other equations, and also carry out the necessary parametric analysis. The software was tested on a classical model of a thin plastic zone (with $s = \text{const}$), for which an exact analytic solution is available.

4. We used the following basic data for the calculation: Young's modulus $E = 200$ GPa, the yield points and specific fracture work for the undamaged material $s_0^+ = s_0^- = 500$ MPa, $\gamma = 15$ kJ/m^2, and the stresses characterizing the resistance of the material to microdamage accumulation were taken as $\sigma_\omega = 5$ GPa and $\Delta\sigma_{th} = 0.25$ GPa. The exponents in (3.4)–(3.6) were taken to be as follows: $m = 4$, $n = 0$, $\alpha = 1$ and $\beta = 4$. In (3.5) we assumed that $\psi_* = 1$ and $c_* = 0.5$. Unless otherwise stated, the loading parameter was taken as $\Delta\sigma_{th} = 150$ MPa for $R = 0.5$, and the initial size of the crack $a_0 = 1$ mm. To illustrate the results we used the following dimensions: the stresses are in MPa, the crack length is in mm, the crack growth rate is in mm/cycle, and the range of the stress intensity factors is in MPa m$^{1/2}$.

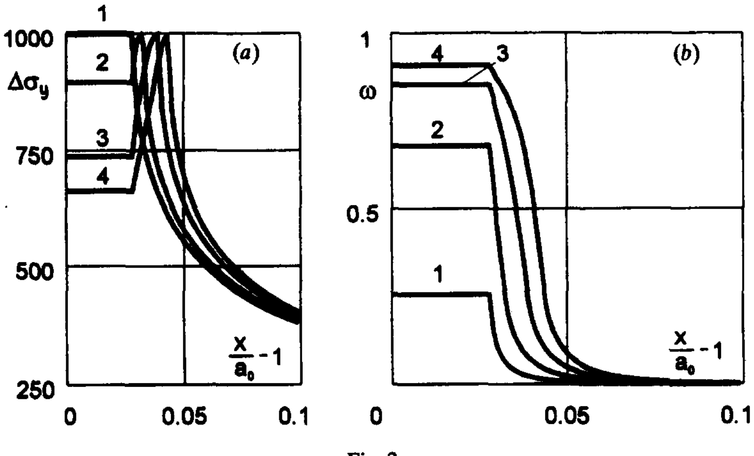

Fig. 2.

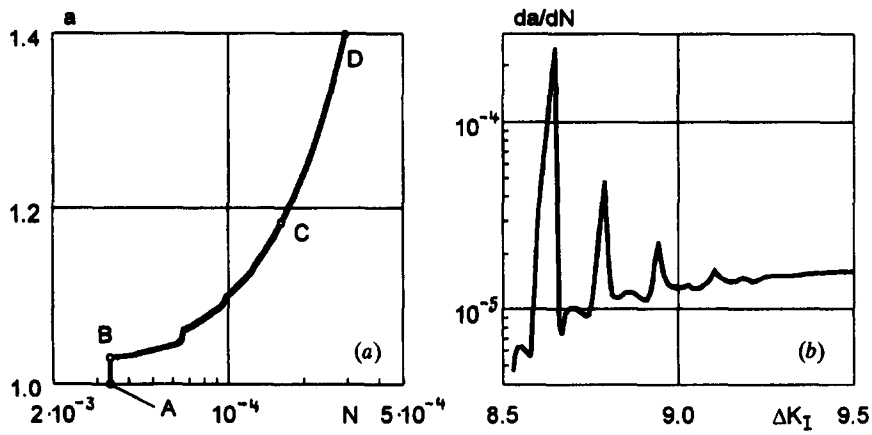

Fig. 3.

In Fig. 2 we show the distribution of the ranges of the stresses $\Delta\sigma_y$ and the measure of microdamage ω in the incubation stage, i.e. from the beginning of loading to the instant when the crack starts. During this stage the stresses in the plastic zone fall, while the damage grows. The crack starts to grow when $N_* = 3305$, while curves 1–3 in Fig. 2 are drawn with a step of $\Delta N = 10^3$, beginning from $N = 500$ (curve 1). Curve 4 corresponds to N_*. On the curves one can clearly see the limits of the cyclic plastic zone, the reduction in the stresses and the growth of microdamage in this zone as the time approaches the instant when the crack begins to grow.

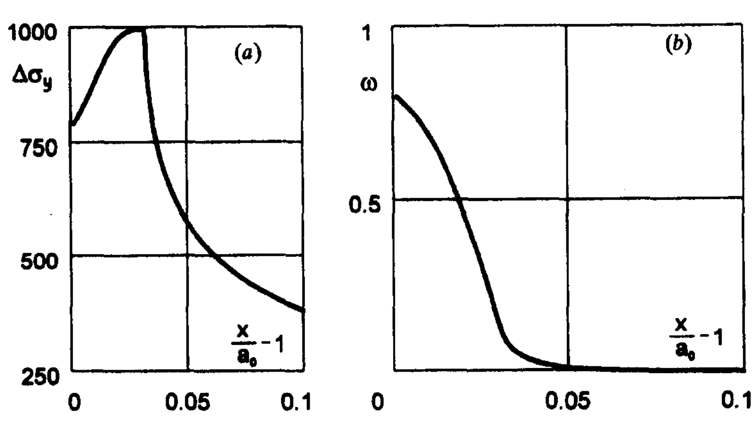

Fig. 4.

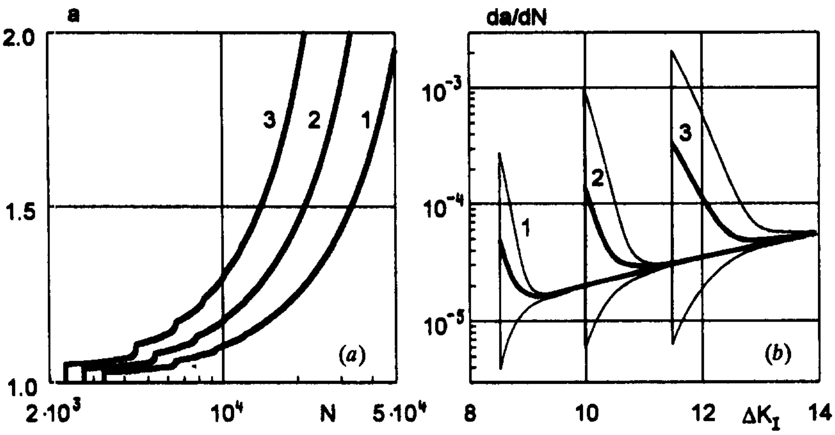

Fig. 5.

In Fig. 3 we show the change in the size of the crack at the initial stage of its propagation. The initial growth of the crack is accompanied by a jump (section AB). Section BC corresponds to the transient stage of crack growth, when the growth rate is an oscillating function of the number of cycles. The crack grows continuously, but its growth alternately accelerates and slows down due to a change in the nature of the microdamage distribution ahead of the front. Hence, along the initial part $a(N)$ is a non-smooth function, while the crack growth rate da/dN is a strongly oscillating function of N. This can be seen both in Fig. 3(a) and (b), where we show the initial part of the diagram of crack growth, i.e. the relationship between the rate da/dN and the range ΔK_{I} of the stress intensity factor $K_I = \sigma_\infty(\pi a)^{1/2}$. Further crack growth (section CD) occurs with a monotonic increase in the rate. When $\Delta K_I > 10$ MPa m$^{1/2}$ the growth rate (on a logarithmic scale) lies practically on a straight line.

The distributions of $\Delta\sigma_y$ and ω, typical for the section CD, are shown in Fig. 4(a) and (b), respectively. Here $N \approx 5 \times 10^4$. The range of stresses $\Delta\sigma_y$ varies monotonically within the tip zone, increasing from $\Delta\sigma_y = 800$ MPa at $x = a$ to $\Delta\sigma_y = 1000$ MPa at $x = a + \lambda_p$. In the elastic zone the range of stresses decreases monotonically, approaching $\Delta\sigma_\infty = 150$ MPa as $x \to a$. The measure of damage varies monotonically ahead of the crack front, leading to continuous advance of the crack.

The effect of the loading level on crack growth is illustrated in Fig. 5. Curves 1, 2 and 3 in Fig. 5(a) are drawn for a range of applied stresses $\Delta\sigma_\infty = 150$, 175 and 200 MPa. The form of the curves changes only slightly over the range of $\Delta\sigma_\infty$ considered. However, for large values of $\Delta\sigma_\infty$ the duration of unsteady crack growth increases. The curves for the rate da/dN as a function of ΔK_{I} diverge strongly at the initial stage, and then approach a common straight line (on a logarithmic scale). In Fig. 5(b) we show averaged values of da/dN, and also the limits of their variation. Details of the variation of the crack growth rate are omitted (they are similar to those shown in Fig. 3b).

Two features of the pattern of crack growth should be noted. First, there is a considerable spread in the rates at the initial stage; the ranges of variation of the rates corresponding to different loading levels overlap. Second, an excess of the averaged rates at the initial stage compared with the values which can be extrapolating the middle

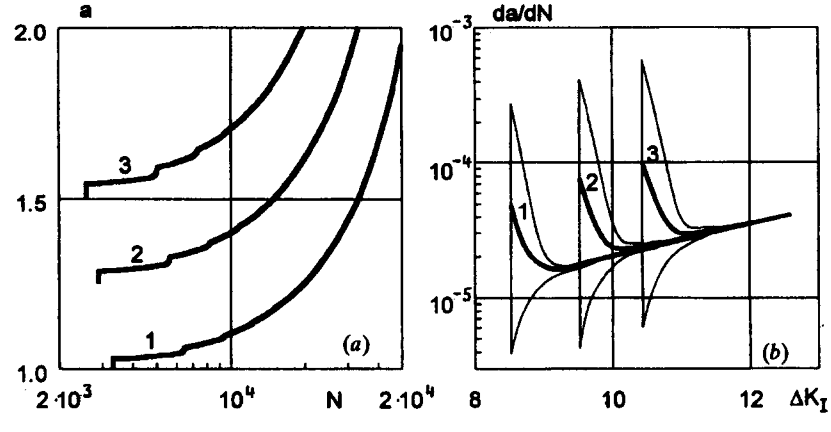

Fig. 6.

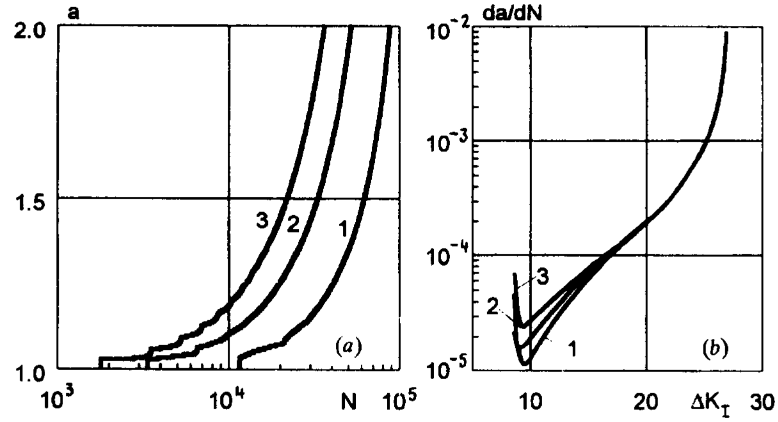

Fig. 7.

section into the region of small ΔK_1 is observed. This effect was also found in simulation using the classical model of a thin plastic zone [12].

Note that in the numerical examples considered the critical value of the degree of fracture resistance for an undamaged material is $K_{IC} = (\gamma E)^{1/2} \approx 55$ MPa m$^{1/2}$. This is much greater than the value $K_I^{max} = \Delta K_I(1 + R)$ shown in Figs 3 and 5 and later. The large spread in the crack growth rate for relatively small ΔK_1 and the anomalous behaviour of short cracks are well-known experimental facts [21]. This has been explained by the "crack closure" effect and related phenomena. It follows from the above analysis that the features of the behaviour of short cracks can also be explained by taking into account the accumulation of damage in the plastic zone.

The above conclusion is also confirmed by an analysis of the effect of the initial crack length on the process of its tip advancement, which is illustrated in Fig. 6. Curves 1, 2 and 3 for the crack length and growth rate drawn for initial lengths of $a_0 = 1$, 1.25 and 1.5 mm, respectively. The calculation was carried out for a loading level of $\Delta\sigma_\infty = 150$ MPa and the material characteristics employed above. Note that the initial dimensions of the crack vary over a very narrow range. Nevertheless, the spread in the crack growth rates at the initial stage is extremely large. For example, by changing the initial crack size by 0.25 mm one can obtain a change in da/dN by an order of magnitude over a small range of ΔK_1. However, as in the previous case, all three curves for the growth rate after relaxation to the steady state lie on a single straight line on a logarithmic scale.

Figure 7, which shows the effect of the degree of cyclic softening on the fatigue-crack growth, contains additional information. Figure 7(a) is drawn for the initial stage of the crack growth while Fig. 7(b) is drawn for the whole range of variation of the rate of advancement of the crack tip. The calculations were carried out for $\Delta\sigma_\infty = 150$ MPa, $R = 0.5$, $\beta = 4$ and three values of the parameter c_* in formula (3.6): $c_* = 0.8$ (curve 1), $c_* = 0.5$ (curve 2) and $c_* = 0$ (curve 3). The last value corresponds to the case when there is no cyclic softening, i.e. $s = s_0 =$ const.

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